where ρ denotes the rank (over the field of rational functions) of $G(s)$, while $\epsilon_i(s)$ and $\psi_i(s)$ are monic and coprime.

Lemma: The blocking polynomial $\beta(s)$ of $G(s)$ is given by $\beta(s) = \epsilon_1(s)$.

Proof: Write

$$G(s) = \frac{1}{d(s)} \cdot N(s)$$

where $N(s)$ is a polynomial matrix and $d(s)$ is the least common multiple (lcm) of the denominators of $G(s)$.

Let $p_i(s)$, $i \in \rho$, denote the invariant polynomials of $N(s)$. Then ([10, p. 139]) $p_1(s)$ is the monic gcd of all entries of $N(s)$. Now since $d(s)$ has no factor that is common to all entries of $N(s)$, it is clear that $p_1(s) = \beta(s)$.

Now ([3, p. 109])

$$\frac{p_i(s)}{d(s)} = \frac{\epsilon_i(s)}{\psi_i(s)}, \qquad i \in \rho$$

and so

$$\frac{\beta(s)}{d(s)} = \frac{\epsilon_1(s)}{\psi_1(s)}.$$

Since $\beta(s)$ and $d(s)$ are coprime, the proof is complete.

From the lemma, the blocking zeros are seen to coincide with the roots, counting multiplicities, of $\epsilon_1(s) = 0$. For the sake of comparison, we remark that Rosenbrock [3] defines the zeros of $G(s)$ to be the roots of $\epsilon_1(s)\epsilon_2(s) \cdots \epsilon_\rho(s) = 0$. Desoer and Schulman [1] and Wolovich [2], without specifying multiplicities, identify the roots of $\epsilon_\rho(s) = 0$ as the zeros of $G(s)$. It is important to note that the blocking zeros introduced *differ*, in general, both in *value* and in *multiplicity* from these other types of zeros. Thus, for the example given in (2) the Smith–McMillan form can be calculated to be

$$M(s) = \begin{bmatrix} \dfrac{s(s^2+1)}{(s+1)^{10}} & 0 & 0 \\ 0 & \dfrac{s^3(s^2+1)^2(s^2+3)}{(s+1)^{10}} & 0 \end{bmatrix}$$

and so the zeros of $G(s)$ according to Rosenbrock are $0, 0, 0, 0, \pm j, \pm j, \pm j, \pm\sqrt{3}\, j$. The zeros defined by Desoer and Schulman (and also by Wolovich) are also located at $0, \pm j, \pm\sqrt{3}\, j$ (multiplicity is not counted). The blocking zeros, however, are $0, \pm j$; $\pm\sqrt{3}\, j$ are *not* blocking zeros.

It is easy to see from the above discussion that the blocking zeros are a subset of the zeros defined in [3] and [4], of the transmission zeros defined in [7] and [11], and of the invariant zeros defined in [11].

III. Concluding Remarks

Desoer and Schulman [1] have shown that if α is a zero of the transfer function matrix, then there exists an m-vector $g \neq 0$ such that if the input of the system is $u(s) = 1/(s-\alpha)g$, the mode $e^{\alpha t}$ will not appear in the output. We have shown that if α is a blocking zero, the vector g is completely arbitrary.

Multivariable system zeros, in general, play an important role in asymptotic tracking problems. The importance of blocking zeros in the context of Davison's problem (see, e.g., [9]) is that in order to attain asymptotic tracking it is necessary that the compensator be designed so that the blocking polynomial of the resultant error transfer function be *divisible* by the characteristic polynomial of the signal (and disturbance) modes. This fact could perhaps be used to further systematize the design of tracking systems.

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An Eigenvalue Characterization of Multivariable System Zeros

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***Abstract*—A characterization of multivariable system zeros as the common eigenvalues of certain matrices is obtained. Major differences are noted between this and other approaches, including implementation and interpretation of results. In addition, a useful determinantal identity is proven.**

Introduction

Determination of zeros via an s-domain representation is epitomized by the process of obtaining Smith–McMillan forms. Such a process is cumbersome for all but the simplest of systems. With an eye towards exploiting computer methods state-space characterization is more convenient for obtaining numerical results.

Consider the system described by

$$\dot{x} = A_0x + B_0u, \quad y = C_0x + D_0u \tag{1}$$

where x is an nth-order state vector, u is an mth-order input vector, y is an rth-order output vector, and the matrices are of appropriate dimensions. The problem of calculating the zeros of (1) has recently been approached in the state-space domain.

Among the proposed alternatives are those presented as follows.

1) Bengtsson (as reported in [1]), who proposes "computing the eigenvalues which are associated with those eigenvectors of $(A + BL_M)$ which lie within the kernel of C," where L_M is a state feedback gain matrix chosen so as to maximize the set of closed-loop eigenvectors lying in the kernel of C.

2) Davison and Wang [2], in which they propose computing the eigenvalues of the matrix

$$S^*(\gamma) = \begin{bmatrix} A_0 - \lambda I & B \\ C & D - \dfrac{\lambda}{\gamma} I \end{bmatrix}$$

where B, C, D are extensions of B_0, C_0, D_0, respectively, and γ is a large real valued scalar.

3) Kouvaritakis (as reported in [1]), who considers (for $D \equiv 0$) the eigenvalues of the closed-loop matrix $(A + BKC)$ as $\|K\| \to \infty$. Davison and Wang [2] also exploit this notion and suggest an alternative computation to 2) based on computing the eigenvalues of $(A + \rho BKC)$ where K is an arbitrary output feedback matrix of full rank and ρ is a scalar to be selected suitably large.

The approach presented in this work uses the problem formulation of Davison and Wang to derive some results of Kouvaritakis. A number of problem areas are resolved via this analysis. Davison and Wang [2] define a system zero as any complex scalar λ^* for which

$$S(\lambda^*) = \begin{bmatrix} A_0 - \lambda^* I & B_0 \\ C_0 & D_0 \end{bmatrix}$$

Manuscript received March 22, 1976; revised May 5, 1976.

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has rank less than $\{n+\min(m,r)\}$. Proper choice of B, C, and D ensures that $S(\cdot)$ is a submatrix of $S^*(\cdot)$ and that $S^*(\cdot)$ is square. Furthermore, for large γ, a subset of the eigenvalues of $S^*(\gamma)$ approach the zeros of $S(\lambda)$. Implementation of this procedure requires that a large numeric value be chosen for γ. The approach offered in the present paper circumvents the necessity of choosing a large γ. This is done by relegating the limiting process to a role which is strictly analytical in nature.

The prime motivation behind the approach offered by Kouvaritakis (as reported in [1]) is the observation that for high feedback gains, the closed-loop poles of multivariable systems approach the open-loop zeros [3]. The results can be stated in two parts depending upon the rank of the product matrix of $[C_0B_0]$. If $[C_0B_0]$ has full rank, then the results of this approach are the same as presented in part 1 of Theorem 1 below. If $[C_0B_0]$ has less than full rank, then the situation is far more complex and the results of Kouvaritakis as reported in [1] are incomplete. In [7], [8], Kouvaritakis and MacFarlane provide the necessary extension and their results are equivalent to part 2 of Theorem 1 below. The principal contribution of this note is a more compact development of these results based on an elaboration of the device of Davison and Wang described above.

The main results of this correspondence are highly dependent upon a determinantal identity presented in Theorem 2. This identity is primarily useful in expressing a special class of determinants in terms of the eigenvalue problem.

The remainder of this correspondence is devoted to presentation and proof of the main results.

Main Results

Consider the problem of determining the zeros of the minimal system defined by (1). Rosenbrock [4] has shown that this can be done as follows:

1) Define the system matrix

$$P(s)=\begin{bmatrix} A_0-sI_n & B_0 \\ C_0 & D_0 \end{bmatrix}.$$

2) Define the $(n+\kappa)\times(n+\kappa)$ minors

$$|P_{i_\kappa,j_\kappa}|=\det\begin{bmatrix} A_0-sI_n & B_{j_\kappa} \\ C_{i_\kappa} & D_{i_\kappa,j_\kappa} \end{bmatrix} \tag{2}$$

where κ is the largest integer such that a nonzero $|P_{i_\kappa,j_\kappa}|$ exists, and i_κ,j_κ denote one set of all possible sets of κ rows of C_0 and D_0, and κ columns of B_0 and D_0, respectively.

3) The zeros of the system $P(s)$ are the zeros of the monic greatest common divisor of all the nonzero minors $|P_{i_\kappa j_\kappa}|$, i.e., those zeros that are common to all of these minors.

It is clear that the crux of the problem lies in the need to find the zeros of a collection of $(n+\kappa)\times(n+\kappa)$ minors of the form described in (2). A solution to this problem is given by the following.

Let the $(n+\kappa)\times(n+\kappa)$ system be defined by

$$S_1(s)=\begin{bmatrix} A-sI_n & B \\ C & D \end{bmatrix}$$

where B and C have full column and row rank κ, respectively. In general, i.e., $D\neq 0$, there exist unimodular matrices such that [5, p. 40]

$$S(s)=\begin{bmatrix} I_n & 0 \\ 0 & R_1 \end{bmatrix} S_1(s) \begin{bmatrix} I_n & 0 \\ 0 & R_2 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} n & \mu & \kappa_1 \end{array} \\ \begin{bmatrix} A-sI_n & B_1 & B_2 \\ C_1 & D_1 & 0 \\ C_2 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ n \\ \mu \\ \kappa_1 \end{array}$$

where $R_1DR_2=\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$ and D_1^{-1} exists (if $D\not\equiv 0$);

$$[BR_2]=[B_1\ \ B_2],\quad [R_1C]'=[C_1'\ \ C_2']$$

$\kappa=\mu+\kappa_1$, and μ and κ_1 are not simultaneously zero.

For any $n\times\kappa$ matrix B and $\kappa\times n$ matrix C, $n\geqslant\kappa$, the following definitions are made. Let C^* be a right inverse of C and B^* a left inverse of B. The matrices Λ_B, Ψ_B, Θ_B, and Λ_C, Ψ_C, Θ_C are defined as follows:

Λ_B is the $n\times(n-\kappa)$ matrix composed of the $n-\kappa$ linearly independent columns of $[I-BB^*]$

Λ_C is the $(n-\kappa)\times n$ matrix composed of the $n-\kappa$ linearly independent rows of $[I-C^*C]$

$$\Psi_B=\begin{bmatrix} B & \Lambda_B \end{bmatrix},\quad \Psi_C=\begin{bmatrix} C \\ \Lambda_C \end{bmatrix},$$

$$\Psi_B^{-1}=\Theta_B=\begin{bmatrix} \Theta_{B1} \\ \Theta_{B2} \end{bmatrix}\begin{array}{l} \kappa \\ n-\kappa \end{array} \qquad \Psi_C^{-1}=\Theta_C=\begin{array}{c}\begin{bmatrix} \Theta_{C1} & \Theta_{C2} \end{bmatrix} \\ \begin{array}{cc} \kappa & n-\kappa \end{array}\end{array}.$$

Theorem 1: The zeros of the matrix $S_1(s)$, defined above, may be obtained as follows.

1) If $\operatorname{rank}[C_2B_2]=\kappa_1$, the zeros of $S_1(s)$ are the roots of

$$|A_4-sI_{n-\kappa_1}|=0 \tag{3}$$

where

$$\Psi_B^{-1}[A-B_1D_1^{-1}C_1]\Psi_C^{-1}= \begin{array}{c} \begin{array}{cc} \kappa_1 & n-\kappa_1 \end{array} \\ \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \end{array} \begin{array}{l} \\ \kappa_1 \\ n-\kappa_1 \end{array}.$$

2) If $\operatorname{rank}[C_2B_2]=\nu<\kappa_1$, the zeros of $S_1(s)$ are the zeros of the reduced matrix

$$\begin{bmatrix} \bar{A}-sI_{n-2\kappa_1+\nu} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$$

where

$$\begin{array}{c} \\ n-2\kappa_1+\nu \\ \kappa_1-\nu \end{array} \begin{array}{c} \begin{array}{cc} n-2\kappa_1+\nu & \kappa_1-\nu \end{array} \\ \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \end{array} = U_1A_4U_2 \tag{4}$$

and U_1, U_2 are unimodular transformations such that

$$U_1(\Theta_{B_22}\Theta_{C_22})U_2=\operatorname{diag}(I_{n-2\kappa_1+\nu}, 0_{\kappa_1-\nu,\kappa_1-\nu}).$$

Special cases: 1) $\kappa_1=0$, i.e., $D=D_1$ The rank test on $[C_2B_2]$ is eliminated and

$$A_4\equiv[A-BD^{-1}C].$$

2) $\mu=0$, i.e., $D\equiv 0$.

The criteria above are computed by replacing

$$B_2\to B, C_2\to C \quad\text{and}\quad [A-B_1D_1^{-1}C_1]\to A.$$

Thus, the zeros of a system have been characterized in (3) as eigenvalues of a matrix.

The following theorem provides the key element in the proofs presented in the next section.

Theorem 2: Let A_0, B_0, and C_0 be dimensioned as in (1). Then for any scalar $\lambda\neq 0$,

$$|A_0|^{(m-1)}\det\left\{A_0+\frac{1}{\lambda}B_0C_0\right\}=\det\left\{I_m|A_0|+\frac{1}{\lambda}C_0(\operatorname{adj}A_0)B_0\right\}. \tag{5}$$

Proofs: Proof of Theorem 2 relies upon Theorem 3.

Theorem 3: Let B_0 and C_0 be dimensioned as in (1). Then for any scalar $\lambda\neq 0$ $(m=r)$

$$\lambda^{(m-n)}|\lambda I_n+B_0C_0|=|\lambda I_m+C_0B_0|. \tag{6}$$

The proof is given by Plotkin [6].

Proof of Theorem 2: Choose a scalar ϵ (possibly zero) so that

$$|A_{01}|=|A_0+\epsilon I_n|\not\equiv 0.$$

Then

$$\det\{A_{01}A_{01}^{-1}\}\det\{\lambda A_{01}+B_0C_0\}=\det\{A_{01}|\lambda I_n+A_{01}^{-1}B_0C_0|\}$$

thus,

$$\lambda^n|A_{01}+\frac{1}{\lambda}B_0C_0|=|A_{01}|\lambda^{(n-m)}\det\{\lambda I_m+C_0A_{01}^{-1}B_0\} \tag{7}$$

where the right-hand side of (7) yields (5) upon noting that

$$A_{01}^{-1}=\frac{\operatorname{adj}A_{01}}{|A_{01}|}$$

and

$$\lim_{\epsilon\to 0}A_{01}=A_0.$$

This completes the proof.

The proof of Theorem 1 is presented first for the case $D\equiv 0$, and then generalized.

Proof of Theorem 1: For $D\equiv 0$, define (for $\epsilon\neq 0$)

$$S(\epsilon)=\begin{bmatrix} A-sI_n & B\\ C & \epsilon I_\kappa\end{bmatrix}$$

and note that

$$\lim_{\epsilon\to 0}S(\epsilon)=S(s)$$

and

$$\lim_{\epsilon\to 0}|S(\epsilon)|=|\lim_{\epsilon\to 0}S(\epsilon)|=|S(s)|.$$

Consider

$$\begin{aligned}\lim_{\epsilon\to 0}|S(\epsilon)|&=\lim_{\epsilon\to 0}\left\{|A-sI_n|\det\left[\epsilon I_\kappa-C(A-sI_n)^{-1}B\right]\right\}\\ &=\lim_{\epsilon\to 0}\left\{\epsilon^\kappa\det\left[I_\kappa|A-sI_n|\right.\right.\\ &\qquad\left.\left.-\frac{1}{\epsilon}C\operatorname{adj}(A-sI_n)B\right]|A-sI_n|^{(1-\kappa)}\right\}\\ &=\lim_{\epsilon\to 0}\left\{\epsilon^\kappa\det\left[\left(A-\frac{1}{\epsilon}BC\right)-sI_n\right]\right\}\end{aligned} \tag{8a}$$

where the last equality follows from (5).

Now note that

$$\begin{aligned}&\det\left[\left(A-\frac{1}{\epsilon}BC\right)-sI_n\right]\cdot|\Psi_B^{-1}|\cdot|\Psi_C^{-1}|\\ &=\det\left\{\Psi_B^{-1}\left[\left(A-\frac{1}{\epsilon}BC\right)-sI_n\right]\Psi_C^{-1}\right\}\\ &=\det\begin{bmatrix}\left(A_1-\frac{1}{\epsilon}I_\kappa-s\Theta_{B1}\Theta_{C1}\right) & A_2-s\Theta_{B1}\Theta_{C2}\\ A_3-s\Theta_{B2}\Theta_{C1} & (A_4-s\Theta_{B2}\Theta_{C2})\end{bmatrix}\\ &=\epsilon^{-\kappa}|\Omega|\det\left[(A_4-s\Theta_{B2}\Theta_{C2})-\epsilon\hat{A}_3\Omega^{-1}\hat{A}_2\right]\end{aligned}$$

where

$$\Omega=[-I_\kappa+\epsilon(A_1-s\Theta_{B1}\Theta_{C1})]$$

$$\hat{A}_2=A_2-s\Theta_{B1}\Theta_{C2},\quad \hat{A}_3=A_3-s\Theta_{B2}\Theta_{C1}.$$

Consequently,

$$\lim_{\epsilon\to 0}S(\epsilon)=\det[(A_4-s\Theta_{B2}\Theta_{C2})]\cdot c,\qquad c=\frac{|\Omega|}{|\Psi_B^{-1}|\cdot|\Psi_C^{-1}|}.$$

Observe that the $n-\kappa$ row space of Θ_{B2} is orthogonal to the κ dim column space of B, and the $n-\kappa$ dim column space of Θ_{C2} is orthogonal to the κ dim row space of C. It follows that the rank deficiency of $\Theta_{B2}\Theta_{C2}$ must be the same as the rank deficiency of CB. Therefore, if $\operatorname{rank}(CB)=\nu\leqslant\kappa$, there exist unimodular transformations U_1, U_2 such that

$$U_1(\Theta_{B2}\Theta_{C2})U_2=\operatorname{diag}(I_{n-2\kappa+\nu},0_{\kappa-\nu,\kappa-\nu}).$$

In conclusion,

$$\lim_{\epsilon\to 0}S(\epsilon)=\det\begin{bmatrix}(\bar{A}-sI_{n-2\kappa+\nu}) & \bar{B}\\ \bar{C} & \bar{D}\end{bmatrix}\cdot c$$

where

$$\begin{array}{ccc} U_1A_4U_2= \begin{bmatrix}\bar{A} & \bar{B}\\ \bar{C} & \bar{D}\end{bmatrix} & \begin{array}{c} n-2\kappa+\nu\\ \kappa-\nu\end{array} & .\\ \quad\;\; n-2\kappa+\nu \quad \kappa-\nu & & \end{array}$$

Thus proving the theorem for $D\equiv 0$.

In a completely analogous manner the proof for the general case $D\not\equiv 0$ is established. The definition of $S(\epsilon)$ is as follows:

$$S(\epsilon)=\begin{bmatrix}A-sI_n & B_1 & B_2\\ C_1 & D_1 & 0\\ C_2 & 0 & \epsilon I_{\kappa_1}\end{bmatrix}$$

in which case (8a) becomes

$$\lim_{\epsilon\to 0}S(\epsilon)=\lim_{\epsilon\to 0}\left\{\epsilon^{\kappa_1}|D_1|\det\left[\left(A-B_1D_1^{-1}C_1-\frac{1}{\epsilon}B_2C_2\right)-sI_n\right]\right\}. \tag{8b}$$

The remaining steps follow by using direct association of (8b) with (8a).

The case of D nonsingular does not involve the limiting process at all. Direct expansion of the resulting $S_1(s)$ followed by application of (5) yields the result stated in special case 1).

This completes the proof.

Conclusions

Theorem 1 provides a means for computing the zeros of the matrices $P_{i_kj_k}$ of (2) by presenting this calculation as an eigenvalue problem. Thus, it can be used as the basis for an algorithm to determine the zeros of a minimal system, and which is structured in accordance with Rosenbrock's suggested procedure outlined above.

The procedure presented here avoids the need to deliberately introduce large numbers into the actual computation as is done in [2]. However, the scheme does require the (possible) calculation of eigenvalues of more than one subsystem, whereas in [2], only one such calculation need be made for a system of order at least as large. Since it is also required to find the intersection of sets of eigenvalues, a tolerance must be defined in order to limit the zero candidates further. The final test is the rank test on $P(s)$ upon substituting a zero candidate for s. This latter test is also needed in the algorithm of [2].

The formulation of Davison and Wang [2] is used as a starting point for the proof of Theorem 1 above. This approach leads to a rather compact and unified development of results previously reported by Kouvaritakis and MacFarlane [1], [7], [8].

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On Eigenvectors of the Canonical Matrix for Multiple-Input Controllable Systems

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Abstract—**The explicit forms of the eigenvectors of the canonical matrix for controllable multiinput systems and the transformation matrix achieving the diagonal matrix are presented.**

I. Introduction

The transformation from the single-input controllable canonical matrix (the companion matrix) to the diagonal matrix can be performed using the Vandermonde matrix [4]. This is readily understood since the ith eigenvector of the companion matrix is given by the form $(1,r_i,r_i^2,\cdots,r_i^{n-1})^T$ where r_i is the ith eigenvalue. This transformation is used in many situations: for example, education of control theory, establishing some theorems, modal analysis, and so on.

In recent years, the canonical matrix for multiple-input controllable (multiple-output observable) systems has been discussed [1]–[3]. However, the corresponding transformation matrix has not appeared.

This correspondence shows the explicit forms of the eigenvectors of this canonical matrix and gives the transformation matrix.

II. Main Result

Consider the following canonical matrix [2] for the controllable systems with m inputs

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m}\\ A_{21} & A_{22} & \cdots & A_{2m}\\ \vdots & \vdots & & \vdots\\ A_{m1} & A_{m2} & & A_{mm}\end{bmatrix} \quad A(n\times n), A_{ij}(n_i\times n_j) \tag{1a}$$

where

$$A_{ii}=\begin{bmatrix} 0 & 1 & & \bigcirc \\ & & 1 & \\ & \bigcirc & & \ddots \\ & & & 1 \\ -a^i_{k_i} & & \cdots & -a^i_{k_{i+1}-1}\end{bmatrix}$$

$$A_{ij}=\begin{bmatrix} & \bigcirc & \\ -a^i_{k_j} & \cdots & -a^i_{k_{j+1}-1}\end{bmatrix}, \quad (i\neq j)$$

$$k_1=1,\ k_i=\sum_{j=1}^{i-1} n_j+1 (i\geq 2),\ n=\sum_{i=1}^{m} n_i. \tag{1b}$$

The following lemma is well known.

Lemma 1 [2], [3]: There exist some unimodular matrices $V(s)$ and $U(s)$ such that

$$V(s)(sI-A)U(s)=\begin{bmatrix} I_{n-m} & 0\\ 0 & D(s)\end{bmatrix} \tag{2a}$$

where

$$D(s)=\begin{bmatrix} d_{11}(s), & d_{12}(s), & \cdots & d_{1m}(s)\\ d_{21}(s), & d_{22}(s), & \cdots & d_{2m}(s)\\ \vdots & \vdots & & \vdots\\ d_{m1}(s), & d_{m2}(s), & \cdots & d_{mm}(s)\end{bmatrix} \quad (m\times m)$$

$$\begin{aligned} d_{ii}(s)&=s^{n_i}+a^i_{k_{i+1}-1}s^{n_i-1}+\cdots+a^i_{k_i+1}s+a^i_{k_i}\\ d_{ij}(s)&=a^i_{k_{j+1}-1}s^{n_j-1}+\cdots+a^i_{k_j+1}s+a^i_{k_j}, \quad (i\neq j)\end{aligned} \tag{2b}$$

namely, $(sI-A)$ and $D(s)$ have the same invariant polynomials except $n-m$ ones which are equal to unity.

Here we assume that $(sI-A)$ has p invariant polynomials $q_1(s),q_2(s),\cdots,q_p(s)$ such that they are not equal to unity and $q_{i+1}(s)$ divides $q_i(s)$. Then there exist two unimodular matrices $L(s)$ $(m\times m)$ and $R(s)$ $(m\times m)$ which satisfy

$$L(s)D(s)R(s)=\mathrm{diag}\big(1,\cdots,1,q_p(s),q_{p-1}(s),\cdots,q_1(s)\big)=S(s) \tag{3}$$

where $S(s)$ is the Smith form of $D(s)$. Note that $m\geq p$ [3] and

$$|D(s)|=|sI-A|=q_p(s)\cdot q_{p-1}(s)\cdots q_1(s). \tag{4}$$

At the outset we consider the case $p=1$ and assume r_i $(i=1\sim n)$ to be the eigenvalues of A. Then the following lemma is established.

Lemma 2: Let $w(s)(m\times 1)$ be the last column vector of $R(s)$. Then $w(s)$ satisfies

$$D(r_i)w(r_i)=0, \qquad (\text{for all } r_i). \tag{5}$$

Proof: Since $|L(s)|=\text{constant}\neq 0$, from (3), we obtain $D(r_i)R(r_i)=L^{-1}(r_i)S(r_i)$ and the last column of $L^{-1}(r_i)S(r_i)$ is 0 for all r_i since $q_1(r_i)=|r_iI-A|=0$. Therefore, if the last column vector of $R(s)$ is defined as $w(s)$, this $w(s)$ satisfies (5). Q.E.D.

The following main theorem is then established.

Theorem 1: 1) The eigenvector t_i associated with the eigenvalue r_i is given by

$$t_i=\begin{bmatrix} w_1(r_i)\begin{bmatrix}1\\ r_i\\ \vdots\\ r_i^{n_1-1}\end{bmatrix} \updownarrow n_1\\ \hline \vdots\\ \hline w_m(r_i)\begin{bmatrix}1\\ r_i\\ \vdots\\ r_i^{n_m-1}\end{bmatrix} \updownarrow n_m\end{bmatrix} \tag{6}$$

where $w_j(r_i)$ is the jth component of the vector $w(r_i)$. 2) If all the eigenvalues are distinct,

$$T^{-1}AT=\mathrm{diag}(r_1,r_2,\cdots,r_n) \tag{7}$$

is achieved where

$$T=(t_1,t_2,\cdots,t_n). \tag{8}$$

Manuscript received October 10, 1976.
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